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VERTEX COLLAPSING AND CUT IDEALS

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Abstract. In this work we study how some elementary graph operations (like the disjoint union) and the collapse of two vertices modify the cut ideal of a graph. They pave the way for reducing the cut ideal of every graph to the cut ideal of smaller ones.

To deal with the collapse operation we generalize the definition of cut ideal given in literature, introducing the concepts of edge labeling and edge multiplicity: in fact we state the *non-classical behavior* of the cut ideal. Moreover we show the transformation of the toric map hidden behind these operations.

In 2008, Sturmfels and Sullivant [7] generalized a class of toric ideals which appears in phylogenetics and algebraic statistics [2], via the cut ideals. *Cuts* are a key concept in graph theory and combinatorial optimization and monomial cut ideals have been further studied in [3], [4] and [5]. Geometrically, the cut ideal of a graph G with e edges comes from the cut polytope $\mathrm{Cut}^{\square}(G)$, the convex hull in $\mathbb{R}^e$ of the *cut semimetrics* [1].

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Our first aim is the study of the cut ideal of a graph obtained after identifying two non connected verticies. We call this operation the collapsing and we denote by $G_{1\equiv k}$ the graph obtained after the collapsing of the verticies 1 and k. We proved the following result:

Theorem 3.1 (Collapsing rules)**.** *Let G be a graph and let $\{1,k\} \notin E$; one obtains $I_{G_{1\equiv k}}$ from I_G, using the following rules:*

(kill) *Kill all the elements in I_G having non-feasible variables;*

(substitute) *Substitute r_{p_f}, the variable of R_G, with r_p, variable of $R_{G_{1\equiv k}}$, in the 'surviving' elements of I_G.*

Moreover using only basic linear algebra and elementary combinatorics, we also study the change of the cut variety under some elementary graph operations (relabeling of vertices and edges, change of multiplicity and disjoint union of graphs) and under the *collapse* of two non-connected vertices.

As a first example, let us consider the path graph P_3, with vertices $\{1,2,3\}$ and edges $\{\{1,2\},\{2,3\}\}$. Let us collapse the vertices 1 and 3. The result is a graph with two vertices and one edge, K_2: This edge should be thought of as a *double* edge.

The cut ideal follows exactly the same phenomena and for this reason, in Section 1, we generalize the concept of cut ideal to a graph G with edge multiplicities and with non trivial edge labels. In the *classical* case, the multiplicities of all the edges are set to be one and the labeling is the canonical labeling (the label of the edge $\{i,j\}$ is (i,j)). Otherwise, we are in the *non-classical* case.

We introduce this generalization to understand collapsing like the previous one, but anyway we find results also in this more general setting.

In Section 2, we explain how to tackle the non-classical setting. We study the *affine cut varieties*, $\mathrm{Aff}(-)$, under the elementary operations of clique 0-sum [7] and disjoint union of two graphs $G \sqcup H$:

Theorem 2.1. *Let G be a graph in the classical case.*

0) $\mathrm{Aff}(S_n)$ *is a point for each $n \in \mathbb{N}$.*

1) $\mathrm{Aff}(G) \cong \mathrm{Aff}(G,\sigma)$ *and $I_G = I_{(G,\sigma)}$ for every multiplicity map σ.*

Let G and H be graphs in the non-classical case.

2) $\mathrm{Aff}(G \sqcup H) \cong \mathrm{Aff}(G \#_0 H)$.

3) $\phi_{G\sqcup H} = \phi_G \phi_H$.

4) $\mathrm{Aff}(G \sqcup S_n) \cong \mathrm{Aff}(G)$ *and $I_{G\sqcup S_n} = I'_G \oplus J$, where J is generated only by linear relations.*

5) *An arbitrary binomial lies in $I_{(G\sqcup H)}$ if and only if either it is linear of the form $r_{a\times b} - r_{a\times b^*}$ with a (resp. b) disjoint partitions of the graph G (resp. H)*

or it is non linear with the form

$$r_{a_1 \times b_1} \cdots r_{a_h \times b_h} - r_{a_{h+1} \times b_{h+1}} \cdots r_{a_{2h} \times b_{2h}}, \tag{1}$$

where

$$r_{a_1} \cdots r_{a_h} - r_{a_{h+1}} \cdots r_{a_{2h}} \in I_{(G,l,A,\sigma)}, \tag{2}$$

and

$$r_{b_1} \cdots r_{b_h} - r_{b_{h+1}} \cdots r_{b_{2h}} \in I_{(H,i,B,\rho)}. \tag{3}$$

The point of view, we propose, helps to generalize Theorem 2.1 of [7] and the results in Section 5 of [4].

After defining the collapsed graph $G_{1\equiv k}$, in Section 3, we show the main theorem that gives a pure combinatorial description of the cut ideal of $G_{1\equiv k}$ from the ideal of I_G in two steps: (*kill*), where we delete the generators containing *non-feasible* variables, and (*substitute*), where we modify the name of the variables according to the collapse. The collapse operation we work with is different from the clique i-sum $G\#_i H$ of G and H. It is possible to construct every clique i-sum with a finite number of collapse operations. For this reason, Theorem 3.1 is a generalization of Theorem 2.1 by Sturmfels and Sullivant [7].

The collapse and the disjoint union operations allow to construct every graph from more elementary graphs. We wonder if this holds also for the cut ideals, that is we can reduce the cut ideal of every graph to the cut ideal of simpler ones: The key is to use Theorem 2.1.**5)** and Theorem 3.1 as we show in Example 3.3. Unfortunately it should not be so easy because difficulties arise in controlling the generators of the collapsing cut ideal in term of the ones of I_G as we show in Example 3.4.

Notation: We denote a graph by a pair $G = (V_G; E_G)$, with $V_G = [n]$. An edge e in E_G with endpoints i and j is denoted by $\{i, j\}$. Moreover K_n, P_n, C_n and S_n denote respectively the complete n-graph, the n-path graph, the n-cycle graph and the graph with n isolated vertices graph.

1. The generalized cut ideal. Let Π_n be the set of disjoint unordered partitions $A|B$ of $[n]$, that is $A \cup B = \{1, 2, \ldots, n\}$ and $A \cap B = \emptyset$. $A|B$ is the same partition as $B|A$, but it is useful to stress the order, so we denote by $(A|B)^* = B|A$. We define

$$\mathrm{Cut}_G(A|B) = \{\{i, j\} \in E_G : i \in A \text{ and } j \in B \text{ or } i \in B \text{ and } j \in A\}.$$

Let R_n be $\mathbb{K}[r_{A|B} : A|B \in \Pi_n]$.

Example 1.1. $R_4 = \mathbb{K}[r_{1|234}, r_{2|134}, r_{3|124}, r_{4|123}, r_{12|34}, r_{13|24}, r_{14|23}, r_{1234|\cdot}]$.

Let A be a generic alphabet set. A *labeling* of E (or of the graph G) is a surjective map $l : E \to \mathrm{A}$.

Example 1.2. The graph K_3 could be labelled by the map sending all the edges to $\{a\}$.

There is a *canonical labeling* $c : E \to E$ that maps injectively an edge $\{i, j\} \in E$ to its endpoints (i, j). Changing the brackets we mean that (i, j) is now an element of the alphabet $\mathrm{A} = E$. All the classical definitions are in the canonical labeling.

Let T_A be $\mathbb{K}[s_a^{\pm 1}, t_a^{\pm 1} : a \in \mathrm{A}]$.

Example 1.3. Let G be any graph with only one edge $E_G = \{\{1, 2\}\}$ and with the canonical labeling (so $\mathrm{A} = \{(1, 2)\}$), then $T_E = \mathbb{K}[s_{(1,2)}, s_{(1,2)}^{-1}, t_{(1,2)}, t_{(1,2)}^{-1}]$.

A *multiplicity map* of E (or of the graph G) is a map $\sigma : E \to \mathbb{Z} \setminus \{0\}$. The integer $\sigma(e)$ is called multiplicity of e.

Example 1.4. Any graph G has the *trivial multiplicity map* setting $\sigma(e) = 1$ for each edge e.

Example 1.5. We assign a multiplicity map to K_2 setting $\sigma(1, 2) \in \mathbb{Z} \setminus \{0\}$.

Notation: In the previous example we used $\sigma(1, 2)$ instead of $\sigma(\{1, 2\})$. This simplification is used in all the article.

We define

$$\begin{aligned} \phi_{(G,l,A,\sigma)} : R_n &\to T_{\mathrm{A}}, \\ r_{A|B} &\mapsto \prod_{\{i,j\} \in \mathrm{Cut}_G(A|B)} s_{l(i,j)}^{\sigma(i,j)} \prod_{\{i,j\} \in E_G \setminus \mathrm{Cut}_G(A|B)} t_{l(i,j)}^{\sigma(i,j)}. \end{aligned}$$

Roughly speaking, we send the variable $r_{A|B}$ to the product of variables in T_{A} including $s_{l(i,j)}^{\sigma(i,j)}$ if $A|B$ separates the extremal vertices i and j of the edge $\{i, j\}$ and including $t_{l(i,j)}^{\sigma(i,j)}$ otherwise. This explains the names for s, separated, and t, together.

If the multiplicity map has value in $\mathbb{N}$, then the map $\phi_{(G,l,\mathrm{A},\sigma)}$ has values in $T'_{\mathrm{A}} = \mathbb{K}[s_a, t_a : a \in \mathrm{A}]$.

Example 1.6. In the classical case (that is with the canonical labeling and the trivial multiplicity) the map $\phi_{(G,c,E,1)}$ is

$$\phi_G : R_n \to T'_E,$$

$$r_{A|B} \mapsto \prod_{\{i,j\}\in \mathrm{Cut}(A|B)} s_{(i,j)} \prod_{\{i,j\}\in E_G\setminus \mathrm{Cut}(A|B)} t_{(i,j)}.$$

Observation 1.1. *In the classical case the map ϕ_G determines G uniquely.*

Definition 1.1. *Let (G,l,A,σ) be a labelled graph with multiplicity; The* cut ideal *of G, $I_{(G,l,A,\sigma)}$, is the kernel of the map $\phi_{(G,l,A,\sigma)}$. The* affine cut variety *of G, $\mathrm{Aff}(G,l,A,\sigma)$, is the affine variety with the coordinate ring $\Gamma_{(G,l,A,\sigma)} = {}^{R_n}/_{I_{(G,l,A,\sigma)}}$.*

Example 1.7. We study P_3 **a)** in the classical case; **b)** with trivial multiplicity but with the labeling given in Example 1.2; **c)** with the same labeling but the multiplicity map is $\sigma(1,2) = -\sigma(2,3) = -1$. One has:

$$\begin{aligned}
\phi_{P_3} : \mathbb{K}[r_{1|23}, r_{2|13}, r_{3|12}, r_{123|\cdot}] &\to \mathbb{K}[s_{(1,2)}, s_{(2,3)}, t_{(1,2)}, t_{(2,3)}];\\
\phi_{(P_3,\{a\})} : \mathbb{K}[r_{1|23}, r_{2|13}, r_{3|12}, r_{123|\cdot}] &\to \mathbb{K}[s_a, t_a].\\
\phi_{(P_3,\{a\},\sigma)} : \mathbb{K}[r_{1|23}, r_{2|13}, r_{3|12}, r_{123|\cdot}] &\to \mathbb{K}[s_a^{\pm 1}, t_a^{\pm 1}].
\end{aligned}$$

and

variable	ϕ_{P_3}	$\phi_{(P_3,\{a\})}$	$\phi_{(P_3,\{a\},\sigma)}$
$r_{1\|23}$	$s_{(1,2)}t_{(2,3)}$	$s_a t_a$	$s_a^{-1}t_a$
$r_{2\|13}$	$s_{(1,2)}s_{(2,3)}$	$s_a s_a = s_a^2$	$s_a^{-1}s_a = 1$
$r_{3\|12}$	$t_{(1,2)}s_{(2,3)}$	$t_a s_a$	$t_a^{-1}s_a$
$r_{123\|\cdot}$	$t_{(1,2)}t_{(2,3)}$	$t_a t_a = t_a^2$	$t_a^{-1}t_a = 1$

Thus, we have $I_{P_3} = (r_{1|23}r_{3|12} - r_{123|\cdot}r_{2|13})$, $I_{(P_3,\{a\})} = (r_{1|23} - r_{3|12}) \oplus I_{P_3}$ and $I_{(P_3,\{a\},\sigma)} = (r_{2|13} - 1, r_{123|\cdot} - 1, r_{1|23}r_{3|12} - 1)$. The classical and the non-classical cut ideals are, hence, different. Looking at the cut varieties we get that $\dim(\mathrm{Aff}(P_3)) = 3$, $\dim(\mathrm{Aff}(P_3,\{a\})) = 2$ and $\dim(\mathrm{Aff}(P_3,\{a\},\sigma)) = 1$.

2. The elementary operations.

Since $\phi_{(G,l,A,\sigma)}$ is a toric map then the cut varieties are toric varieties, thus we associate (see for instance [6]) to $\phi_{(G,l,A,\sigma)}$ the matrix $\mathcal{A}_{(G,l,A,\sigma)}$ having as columns the exponents of the monomial image of $r_{A|B}$ for each partition in Π_n. (For sake of brevity we write when it is possible $\mathcal{A}_G$ instead of $\mathcal{A}_{(G,l,A,\sigma)}$.)
The generators of I_G correspond to the elements in the kernel of the linear map defined by $\mathcal{A}_G$.

We want to study which relation there is between elementary operations on the graph and linear transformations of the matrices $\mathcal{A}_G$.

Notation: We remark $G\#_i H$ is the notation for a *clique i-sum* of two graph G and H, that is the gluing of G and H along a specified clique. Moreover, the *disjoint union* of G and H is $(G, l, A, \sigma) \sqcup (H, i, B, \rho) = (G \sqcup H, (l, i), A \sqcup B, (\sigma, \rho))$.

Any disjoint partition of $G \sqcup H$ can be written as $(AC|BD)$ where $a = (A|B)$ is a G-partition and $b = (C|D)$ is a H-partition. So we think it as a product partition $a \times b$. From a and b it is also possible to construct $a \times b^*$ that is different from $a \times b$.

Lemma 2.1. *Let G be a graph in the classical case. Let $E = \{e_1, \ldots, e_m\}$ and $A = \{a_1, \ldots, a_k\}$. Let σ be a multiplicity map and $l : E \to A$ be a labeling map of G. Then*

i) *Permuting the name of vertices corresponds to a permutation of the matrix columns of $\mathcal{A}$.*

ii) *There exists a unique matrix M_σ such that $\mathcal{A}_{(G,\sigma)} = M_\sigma \mathcal{A}_G$. M_σ is a block matrix*

$$M_\sigma = \begin{pmatrix} I_\sigma & 0 \\ 0 & I_\sigma \end{pmatrix}$$

where $I_\sigma = \mathrm{diag}(\sigma(\mathrm{e}_1), \ldots, \sigma(\mathrm{e_m}))$.

iii) *There exists a unique matrix R_l such that $\mathcal{A}_{(G,l,A)} = R_l \mathcal{A}_G$. This matrix is $2|A| \times 2|E|$ and it has the block form*

$$R_l = \begin{pmatrix} B_l & 0 \\ 0 & B_l \end{pmatrix}$$

where $B_l = (b_{i,j})$ is $|A| \times |E|$ and it is defined by $b_{(i,j)} = \delta_{a_i, l(e_j)}$.

Let G and H be graphs in the non-classical case.

iii.bis) *Let l' be a labeling constructed from l by assigning to the elements in $l^{-1}(a_k)$ a unique element in $C = A \setminus \{a_k\}$. Then there exists a unique matrix $R_{l'}$ such that $\mathcal{A}_{(G,l',C)} = R_{l'} \mathcal{A}_G$. This matrix is $2|C| \times 2|A|$ and it has the block form*

$$R_{l'} = \begin{pmatrix} B_{l'} & 0 \\ 0 & B_{l'} \end{pmatrix}$$

where $B_{l'} = (b_{i,j})$ is $|C| \times |A|$ and it is defined by $b_{(i,j)} = \delta_{a_i, l'(a_j)}$.

iv) *$\mathcal{A}_{G \sqcup H}$ is made of the columns of $\mathcal{A}_G \# \mathcal{A}_H$ but each repeated twice.*

v) *$\mathcal{A}_{G \#_0 H} = \mathcal{A}_G \# \mathcal{A}_H$.*

Notation: The columns of $\mathcal{A}_G \# \mathcal{A}_H$ are constructed mixing the columns of $\mathcal{A}_G$ and $\mathcal{A}_H$ in all the possible ways.

Proof. **i)**, **ii)**, **iii)** and **iii.bis)** are elementary. Regarding **iv)**, we observe that for each a and b, respectively G and H-partitions, the $G \sqcup H$-partitions $a \times b$ and $a \times b^*$ separate and leave together the same edges. **v)** holds because whatever pair of vertices $(v, w) \in V_G \times V_H$ we choose for the clique 0-sum $G\#_0 H$, one and only one of those partitions $a \times b$ and $a \times b^*$ leave the pair on one size. □

Using this matrix tricks we get some information about the cut variety:

Theorem 2.1. *Let G be a graph in the classical case.*

0) $\operatorname{Aff}(S_n)$ *is a point for each* $n \in \mathbb{N}$.

1) $\operatorname{Aff}(G) \cong \operatorname{Aff}(G, \sigma)$ *and* $I_G = I_{(G,\sigma)}$ *for every multiplicity map* σ.

Let G and H be graphs in the non-classical case.

2) $\operatorname{Aff}(G \sqcup H) \cong \operatorname{Aff}(G\#_0 H)$.

3) $\phi_{G\sqcup H} = \phi_G \phi_H$.

4) $\operatorname{Aff}(G \sqcup S_n) \cong \operatorname{Aff}(G)$ *and* $I_{G\sqcup S_n} = I'_G \oplus J$, *where J is generated only by linear relations.*

5) *An arbitrary binomial lies in* $I_{(G\sqcup H)}$ *if and only if either it is linear of the form* $r_{a\times b} - r_{a\times b^*}$ *with a (resp. b) disjoint partitions of the graph G (resp. H) or it is non linear with the form*

$$r_{a_1\times b_1} \cdots r_{a_h\times b_h} - r_{a_{h+1}\times b_{h+1}} \cdots r_{a_{2h}\times b_{2h}}, \tag{4}$$

where

$$r_{a_1} \cdots r_{a_h} - r_{a_{h+1}} \cdots r_{a_{2h}} \in I_{(G,l,A,\sigma)}, \tag{5}$$

and

$$r_{b_1} \cdots r_{b_h} - r_{b_{h+1}} \cdots r_{b_{2h}} \in I_{(H,i,B,\rho)}. \tag{6}$$

Proof. ϕ_{S_n} sends all variables of R_n (and 1) to $1 \in T_\emptyset = \mathbb{K}$; thus **0)** holds. **1)** follows from **ii)** and **2)** follows from **iv)** and **v)**. **3)** is **iv)** translated with the homomorphism language.

The first part of **4)** is a consequence of **2)**. For the latter we observe that for any S_n-disjoint partition $(C|D)$, using **3)**, $r_{(A|BCD)}$ has the same image of $r_{(AC|BD)}$ and of all the other possible further combinations. Thus J is generated by those linear relations and I'_G is constructed from I_G by replacing the variable $r_{(A|B)}$ with $r_{(A|BCD)}$. Using **3)** and **iv)**, we obtain **5)**. □

The last item of the previous theorem is a generalization of Theorem 2.1 of [7] and of the results in Section 5 of [4].

Observation 2.1. *Example 1.7 shows that **2)** is not true for a non-classical setting.*

Example 2.1. One has that $I_{K_2} = (0)$ and $I_{K_2\sqcup K_2} = r_{1|234} r_{3|124} - r_{24|13} r_{1234|\cdot}$,
$r_{1234|\cdot} - r_{12|34}, r_{4|123} - r_{3|124}, r_{2|134} - r_{1|234}, r_{13|24} - r_{14|23}$. We have no generator in I_{K_2} to construct $I_{K_2\sqcup K_2}$ using **5)**. Instead, we use binomials like $r_{1|2} - r_{1|2} \in I_{K_2}$.

3. The collapse operation. In this section, we study what happens to the cut ideals and the cut varieties after collapsing two vertices. In the first part, we study the *simple* collapse, and then we will go to the *singular* one. We see how the non-canonical labeling and non-trivial multiplicity appear naturally.

This is the collapse operation:

Definition 3.1. *Let* (G, l, A, σ) *be a graph and let* $\{1, k\} \notin E$*. The* graph obtained by collapsing the vertices 1 and k *is denoted by* $(G_{1\equiv k}, l', A', \sigma')$*. We define* $G_{1\equiv k} = (\{1, \ldots, k-1\}; E')$ *where* E' *is obtained from* E_G *by replacing* $\{i, k\} \in E_G$ *with* $\{i, 1\}$*, and considering just one repetition; the labeling map* l' *is the same as* l*, but for all the edges in* $e \in l^{-1}(l(i, k))$ *we set* $l'(e) = l(i, 1)$*;*

$$\sigma'(\{i, j\}) = \begin{cases} \sigma(\{i, j\}) & \text{if } i \neq 1; \\ \sigma(\{1, j\}) + \sigma(\{k, j\}) & \text{otherwise.} \end{cases}$$

We say that the collapse is simple *if* $|E_G| = |E_{G_{1\equiv k}}|$*, and* singular *otherwise.*

Only for singular collapse we will have that $A' \subsetneq A$: in fact we lose one of the labels of the collapsed edges.

Observation 3.1. *When we write* $(G \sqcup H)_{k\equiv k+1}$ *and* $G\#_0 H$ *we mean the same thing.*

Of course, the collapse is not always a clique 0*-sum (see next example). Instead every clique* i*-sum can be constructed as a sequence of collapses.*

Example 3.1. $K_2 = (P_3)_{1\equiv 3}$. This collapse is singular and it produces the graph with multiplicity given in Example 1.5.

Example 3.2. The singular collapse can involve more than two edges. For example $G_{1\equiv 5}$, where

$$G = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{5, 2\}, \{5, 3\}, \{5, 4\}\}).$$

Definition 3.2. *A disjoint partition* $(A|B)$ *is feasible for the collapse of* 1 *and* k *if* $\{1, k\} \in \mathrm{Cut}_{K_n}(A|B)$*.*

In other words we require that the collapsed vertices belong both to either A or B. If p is a feasible partition then $r_p \in R_n$ is a *feasible variable.* Moreover, let p be a disjoint partition of $[n-1]$, then we denote by p_f and p_{nf} the feasible and the non-feasible lifting to the partitions of $[n]$.

The following theorem shows a pure combinatorial description of the ideal $I_{G_{1\equiv k}}$ from the ideal of I_G. This theorem is stated in the classical and non classical case for simple and singular collapse.

Theorem 3.1 (Collapsing rules)**.** *Let G be a graph and let $\{1,k\} \notin E$; one obtains $I_{G_{1\equiv k}}$ from I_G, using the following rules:*

(kill) *Kill all the elements in I_G having non-feasible variables;*

(substitute) *Substitute r_{p_f}, the variable of R_G, with r_p, variable of $R_{G_{1\equiv k}}$, in the 'surviving' elements of I_G.*

We can construct every graph from some of its subgraphs via disjoint unions and operations of simple collapse. This idea should hold also for the cut ideal: The key is to use Theorem 2.1.**5)** and Theorem 3.1 as we show in the following example. In contrast, in Example 3.4, we stress that Theorem 3.1 does not allow to control the generators of the collapsing cut ideal in terms of the ones of I_G.

Example 3.3. We compute I_{P_4} via the cut ideal of $P_3 \sqcup K_2$, using a suitable vertices collapse. We label the vertices of K_2 with 4 and 5. We saw in Example 2.1 that $I_{P_3} = (r_{1|23}r_{3|12} - r_{2|13}r_{123|\cdot})$ and also that $I_{K_2} = (0)$. We know how to produce the linear relation between the variables (like $r_{a\times b} - r_{a\times b^*}$).

Let us focus on the non linear part. We need to start from an element in I_{P_3}: for instance we have $r_{2|13}r_{3|12} - r_{2|13}r_{3|12} = 0 \in I_{P_3}$. and $r_{4|5}r_{45|\cdot} - r_{4|5}r_{45|\cdot} = 0 \in I_{K_2}$. Thus we compose them into

$$r_{2\mathbf{4}|13\mathbf{5}}r_{3\mathbf{54}|12} - r_{2|13\mathbf{54}}r_{3\mathbf{5}|12\mathbf{4}}$$

obtaining an element of $I_{P_3\sqcup K_2}$. In similar way, by changing only the element in I_{P_3}, one obtains also:

$$\begin{aligned} r_{\mathbf{4}|123\mathbf{5}}r_{12|3\mathbf{45}} &- r_{123\mathbf{45}|}r_{3\mathbf{5}|12\mathbf{4}}, \\ r_{1|23\mathbf{54}}r_{3\mathbf{5}|12\mathbf{4}} &- r_{12|3\mathbf{54}}r_{1\mathbf{4}|23\mathbf{5}}, \\ r_{1|23\mathbf{54}}r_{13\mathbf{5}|2\mathbf{4}} &- r_{2|13\mathbf{54}}r_{1\mathbf{4}|23\mathbf{5}}, \\ r_{1|23\mathbf{54}}r_{\mathbf{4}|123\mathbf{5}} &- r_{123\mathbf{54}|}r_{1\mathbf{4}|23\mathbf{5}}. \end{aligned}$$

Moreover, considering the non zero generator $r_{1|23}r_{3|12} - r_{123|\cdot}r_{2|13}$ of I_{P_3} and $r_{45|\cdot}r_{45|\cdot} - r_{45|\cdot}r_{45|\cdot} \in I_{K_2}$ one has the following element of $I_{P_3\sqcup K_2}$:

$$r_{1|23\mathbf{45}}r_{3\mathbf{45}|12} - r_{123\mathbf{45}|\cdot}r_{2|13\mathbf{45}}.$$

Thus, by changing the element in I_{K_2}, one has:

$$\begin{aligned} r_{\mathbf{4}|123\mathbf{5}}r_{13\mathbf{5}|2\mathbf{4}} &- r_{1\mathbf{4}|23\mathbf{5}}r_{3\mathbf{5}|12\mathbf{4}}, \\ r_{1|23\mathbf{45}}r_{3\mathbf{5}|12\mathbf{4}} &- r_{\mathbf{4}|123\mathbf{5}}r_{2|13\mathbf{45}}, \\ r_{1|23\mathbf{45}}r_{3\mathbf{5}|12\mathbf{4}} &- r_{123\mathbf{45}|\cdot}r_{13\mathbf{5}|2\mathbf{4}}. \end{aligned}$$

This completes the non linear generators of the cut ideal of $I_{P_3\sqcup K_2}$.

We compute I_{P_4} using Theorem 3.1: one collapses the vertices 3 and 5. Thus one has:

$$I_{P_4} = \begin{pmatrix} r_{13|24}r_{12|34} - r_{2|134}r_{3|124}, \\ r_{4|123}r_{12|34} - r_{1234|}r_{3|124}, \\ r_{1|234}r_{3|124} - r_{12|34}r_{14|23}, \\ r_{1|234}r_{13|24} - r_{2|134}r_{14|23}, \\ r_{1|234}r_{4|123} - r_{1234|}r_{14|23}, \\ r_{4|123}r_{13|24} - r_{14|23}r_{3|124}, \\ r_{1|234}r_{3|124} - r_{4|123}r_{2|134}, \\ r_{1|234}r_{3|124} - r_{13|24}r_{1234|\cdot}, \\ r_{1|234}r_{12|34} - r_{1234|}r_{2|134} \end{pmatrix}.$$

Example 3.4. Let

$$G = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{4,5\}\}).$$

Its cut ideal is

$$I_G = \begin{pmatrix} r_{13|245}r_{4|1235} - r_{123|45}r_{24|135}, r_{12|345}r_{24|135} - r_{13|245}r_{34|125}, \\ r_{12|345}r_{4|1235} - r_{123|45}r_{34|125}, r_{12|345}r_{13|245} - r_{2|1345}r_{3|1245}, \\ r_{124,35}r_{2|1345} - r_{12|345}r_{24|135}, r_{124,35}r_{123|45} - r_{3|1245}r_{4|1235}, \\ r_{124,35}r_{13|245} - r_{3|1245}r_{24|135}, r_{124,35}r_{12|345} - r_{3|1245}r_{34|125}, \\ r_{14|235}r_{3|1245} - r_{124,35}r_{23|145}, r_{14|235}r_{123|45} - r_{23|145}r_{4|1235}, \\ r_{14|235}r_{13|245} - r_{23|145}r_{24|135}, r_{14|235}r_{12|345} - r_{23|145}r_{34|125}, \\ r_{134|25}r_{23|145} - r_{14|235}r_{2|1345}, r_{134|25}r_{3|1245} - r_{12|345}r_{24|135}, \\ r_{134|25}r_{123|45} - r_{2|1345}r_{4|1235}, r_{134|25}r_{13|245} - r_{2|1345}r_{24|135}, \\ r_{134|25}r_{12|345} - r_{2|1345}r_{34|125}, r_{134|25}r_{124,35} - r_{34|125}r_{24|135}, \\ r_{5|1234}r_{2|1345} - r_{134|25}r_{12345|\cdot}, r_{5|1234}r_{3|1245} - r_{124,35}r_{12345|\cdot}, \\ r_{5|1234}r_{123|45} - r_{12345|\cdot}r_{4|1235}, r_{5|1234}r_{13|245} - r_{12345|\cdot}r_{24|135}, \\ r_{5|1234}r_{12|345} - r_{12345|\cdot}r_{34|125}, r_{5|1234}r_{14|235} - r_{15|234}r_{4|1235}, \\ r_{1|2345}r_{4|1235} - r_{5|1234}r_{23|145}, r_{1|2345}r_{4|1235} - r_{14|235}r_{12345|\cdot}, \\ r_{1|2345}r_{34|125} - r_{12|345}r_{15|234}, r_{1|2345}r_{24|135} - r_{13|245}r_{15|234}, \\ r_{1|2345}r_{4|1235} - r_{123|45}r_{15|234}, r_{1|2345}r_{123|45} - r_{12345|\cdot}r_{23|145}, \\ r_{1|2345}r_{124,35} - r_{3|1245}r_{15|234}, r_{1|2345}r_{14|235} - r_{23|145}r_{15|234}, \\ r_{1|2345}r_{134|25} - r_{2|1345}r_{15|234}, r_{1|2345}r_{5|1234} - r_{12345|\cdot}r_{15|234} \end{pmatrix}.$$

We collapse the vertices 1 and 5 obtaining K_4. The cut ideal of K_4 is

$$I_{K_4} = (r_{1|234}r_{2|134}r_{3|124}r_{4|123} - r_{1234|\cdot}r_{23|14}r_{12|34}r_{13|24})$$

We want to compute it using the Theorem 3.1. We observe that all the generators of I_G contain at least one of the non feasible variables $r_{1|2345}$, $r_{5|1234}$, $r_{134|25}$, $r_{14|235}$, $r_{124,35}$, $r_{12|345}$, $r_{13|245}$ and $r_{123|45}$; hence all of them will be killed. One has that $r_{15|234}r_{2|1345}r_{3|1245}r_{4|1235} - r_{12345|\cdot}r_{23|145}r_{125|34}r_{135|24}$ is in the cut ideal I_G; this element survives because of contains only feasible elements; moreover the collapsing substitution produces exactly the generator we wanted.

The rest of this section is devoted to the prove of Theorem 3.1. The simple and singular cases are different so we split the proof in two proposition analysing them separately.

3.1. The simple collapse. In this section we study the simple collapse of graphs in the classical and non-classical case. The simple collapse does not change the number of edges or the multiplicities of them and if we start from a graph with trivial multiplicity, then we obtain a graph with trivial multiplicity.

$G\#_0 H = (G \sqcup H)_{k\equiv k+1}$ is an example of simple collapse where the cut varieties are isomorphic. This is not a general fact:

Example 3.5. K_3 could be seen as the collapse of 1 and 4 in P_4. We compute that

$$R_{P_4} = \frac{\mathbb{K}[r_{1|234}, r_{2|134}, r_{3|124}, r_{4|123}, r_{12|34}, r_{13|24}, r_{14|23}, r_{1234|\cdot}]}{I_{P_4}},$$
$$R_{K_3} = \frac{\mathbb{K}[r_{1|23}, r_{2|13}, r_{3|12}, r_{123|\cdot}]}{(0)},$$

where I_{P_4} is generated by the nine quadratic equations in Example 3.3. Hence $\mathrm{Aff}(P_4) \ncong \mathrm{Aff}((P_4)_{(1\equiv 4)})$.

The matrix $\mathcal{A}_G$ implicitly gives an order of the variables of R_n. In what follow we use the letter p to denote the partition of a variable r_p and the letter k to indicate that r_k is the k-th variable in this order.

Lemma 3.1. *Let G be a graph and let $\{1,k\} \notin E$. Let the collapse of 1 and k be simple. Then there exist a finite number of matrices $C_{1\equiv k}$ so that $\mathcal{A}_G C_{1\equiv k} = \mathcal{A}_{G_{1\equiv k}}$. If $\mathcal{A}_G$ is made of the block matrices (F, N) where F (resp. N) is the matrix of the exponents of the image of the feasible (resp. non feasible) variables, then $C_{1\equiv k}$ is a $2^{n-1} \times 2^{n-2}$ matrix and it has the following block form*

$$C_{1\equiv k} = \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}$$

P r o o f. By the assumption, $\{1,k\}$ is not an edge, so $T_{A_G} = T_{A_{G_{1\equiv k}}}$. Fixing the collapse we fix the injection $\tilde{(.)} : R_{n-1} \hookrightarrow R_n$ where p is sent to p_f. The collapsing is simple and so one has

$$\phi_{G_{1\equiv k}}(r_p) = \phi_G(r_{p_f}) = \phi_G(\tilde{r_p}). \tag{7}$$

Thus $\phi_{G_{1\equiv k}}$ factors through the composition $R_{n-1} \overset{\tilde{(.)}}{\hookrightarrow} R_n \overset{\phi_G}{\to} T_{A_G}$. The injection map $\tilde{(.)}$ corresponds to the matrix $C_{1\equiv k}$. □

Proposition 3.1. *Theorem* 3.1 *holds for simple collapses.*

P r o o f. Using the previous lemma we know that $\phi_{G_{1\equiv k}}$ factors through the composition $R_{n-1} \overset{\tilde{(.)}}{\hookrightarrow} R_n \overset{\phi_G}{\to} T_{A_G}$. If $x \in R_{G_{1\equiv k}}$ and $\phi_{G_{1\equiv k}}(x) = 0$ then $\phi_G(\tilde{x}) = 0$, where $\tilde{x}$ is the lifting of x in R_n. This prove the (*substitute*) rule. The (*kill*) property follows form the fact that p_{nf} is a lifting that does not correspond to any partition in $G_{1\equiv k}$. □

3.2. The singular collapse. The notions of multiplicity and labeling that we introduced deal with the singular collapse.

We note that before and after a singular collapse the domain changes because the number of vertices change as well; the codomain changes because we decrease the number of the edges.

Lemma 3.2. *Let G be a graph and let $\{1,k\} \notin E$. Let the collapse of* 1 *and k be singular and let the collapsing pairs of edges have the same labels for each pair. Then there exist a finite number of matrices $C_{1\equiv k}$ so that $\mathcal{A}_G C_{1\equiv k} = \mathcal{A}_{G_{1\equiv k}}$ with the same form of Lemma* 3.1.

P r o o f. The proof follows as in Lemma 3.1. (7) holds because the feasible partitions separate or divide, at the same time, the collapsing edges. □

Lemma 3.3. *Let G be a graph and let $\{1,k\} \notin E$. Let the collapse of* 1 *and k be singular. Then there is a finite number of matrices $C_{1\equiv k}$ such that $(R_{l'}\mathcal{A}_G)C_{1\equiv k} = \mathcal{A}_{G_{1\equiv k}}$, where*

- *l' is the labeling such that each collapsing couple of edges has the same labels;*
- *$C_{1\equiv k}$ is the matrix of the collapse as in the previous lemma.*

P r o o f. Without loss of generality we can assume that the singular collapse involves only two edges: $\{1,l\}$ and $\{l,k\}$. Any singular collapse splits in two steps. We work in the non-classical setting, so it is possible that $\{1,l\}$ and $\{l,k\}$ have the same label; if not, we relabel them with the same one, a. We call the new label l'. This gives a labelled graph (G, l', A', σ). Using **iii.bis)** of Lemma 2.1 to the relabeling correspond a unique matrix $R_{l'}$. We collapse the two vertices of (G, l', A', σ). Using the previous lemma, there is a a matrix $C_{1\equiv k}$ controlling the collapse. □ In other words, before we collapse the two vertices in the codomain, that is we let the two edges $(\{1, l\}, \{l, k\})$ be considered as a unique edge in T_A; then we collapse the two vertices in the domain, that is we select the feasible partition of $[n-1]$.

Proposition 3.2. *Theorem* 3.1 *holds for singular collapses.*

Proof. Without lost of generality we assume that the singular collapse involves only two edges: $\{1, l\}$ and $\{l, k\}$. Using the previous lemma we know that $\phi_{(G_{1\equiv k}, l, A, \sigma)}$ factors through

$$R_{n-1} \overset{\tilde{(\cdot)}}{\hookrightarrow} R_n \overset{\phi_G}{\to} T_G \twoheadrightarrow T_{A_{G_{1\equiv k}}}.$$

Let a be the label of the two collapsing edges and let l' be the new labeling. If $r_q \in R_n$, looking at the map ϕ_G, one sees that

$$\phi_{(G,l',A')}(r_q) = \begin{cases} \cdots s_a^{\sigma(1,l)} t_a^{\sigma(l,k)} \cdots & q \text{ separates } \{1, l\} \text{ but not } \{l, k\}; \\ \cdots s_a^{\sigma(l,k)} t_a^{\sigma(1,l)} \cdots & q \text{ separates } \{l, k\} \text{ but not } \{1, l\}; \\ \cdots s_a^{\sigma(1,l)+\sigma(l,k)} \cdots & q \text{ separates both } \{1, l\}, \{l, k\}; \\ \cdots t_a^{\sigma(1,l)+\sigma(l,k)} \cdots & \text{otherwise.} \end{cases} \tag{8}$$

After the collapsing of 1 and k, following the notation of the previous lemma, we get the graph $(G_{1\equiv k}, l', A', \sigma')$. The collapse produces a change of the multiplicity of the edge $\{1, l\}$: $\sigma'(1, l) = \sigma(1, l) + \sigma(l, k)$. Let $r_p \in R_{n-1}$, one has

$$\phi_{(G_{1\equiv k}, l', A', \sigma)}(r_p) = \begin{cases} \cdots s_a^{\sigma(1,l)+\sigma(l,k)} \cdots & p \text{ separates the vertices 1 and } l; \\ \cdots t_a^{\sigma(1,l)+\sigma(l,k)} \cdots & \text{otherwise.} \end{cases}$$

The maps $\phi_{(G,l',A',\sigma)}$ and $\phi_{(G_{1\equiv k}, l', A', \sigma')}$ are coherent: if $p = A|B$ is a $(n-1)$-partition, then $\phi_{(G_{1\equiv k}, \sigma)}(r_p) = \phi_{(G,l,A)}(\tilde{r_p})$, where $\tilde{r_p}$ is the usual lifting of r_p. There are no partitions p of $[n-1]$ such that p' separates only one of the edges $\{1, l\}, \{l, k\}$: this implies that the first case of the equation (8) is not possible after the collapsing of 1 and k. □

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